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Vibrations of Elastically Restrained Nonuniform Beams with Arbitrary Pretwist

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The coupled governing differential equations with general elastic boundary conditions for coupled bending–bending vibration of a pretwisted nonuniform beam are derived by using Hamilton's principle. The field transfer matrix for the general system is expressed in integration form. As the elements of the matrix can be integrated analytically, the exact field matrix is, therefore, obtained. The transfer matrix method is presented to study the vibrations of any nonuniform cross-sectional beam with arbitrary variation of pretwist angle. The frequency equation is derived and expressed in terms of the transfer matrix. Several limiting cases are examined. The corresponding results are compared with previous work in the literature. Finally, the influence of elastically restrained boundary conditions and uniform and nonuniform pretwists on the natural frequencies is investigated.

Nomenclature

$A(x)$	= cross-sectional area of the beam
$B_{ij}(\xi)$	= dimensionless bending rigidity, $E(x)I_{ij}(x)/[E(0)I_{yy}(0)]$, $i, j = x, y$
$E(x)$	= Young's modulus of beam material
$I(x)$	= area moment inertia of the beam
$\bar{K}$	= kinetic energy
$K_{yTL}, K_{y\theta L}, K_{yTR}$, $K_{y\theta R}$ and K_{zTL}, $K_{z\theta L}, K_{zTR}, K_{z\theta R}$	= rotational and translational spring constants at the left and the right end of the beam in the y and z directions, respectively
L	= length of the beam
$m(\xi)$	= dimensionless mass, $\rho(x)A(x)/[\rho(0)A(0)]$
$[T]$	= overall transfer matrix
$[T_f]_j$, $[T_s]_j$	= jth field and station transfer matrix
t	= time variable
$\bar{U}$	= potential energy
$u(x,t)$, $v(x,t)$, $w(x,t)$	= beam lateral displacement in the x, y, and z directions, respectively
$V(\xi)$, $W(\xi)$	= dimensionless lateral displacement in the y and z directions, respectively, v/L and w/L
X, Y, Z	= principal frame coordinates
x, y, z	= fixed frame coordinates
$\beta_1, \beta_2, \beta_3, \beta_4$, $\beta_5, \beta_6, \beta_7, \beta_8$	= dimensionless rotational and translational spring constants at the left and right of the beam in the y and z directions, respectively, $K_{z\theta L}L/[E(0)I_{yy}(0)]$, $K_{zTL}L/[E(0)I_{yy}(0)]$, $K_{y\theta L}L/[E(0)I_{yy}(0)]$, $K_{yTL}L/[E(0)I_{yy}(0)]$, $K_{z\theta R}L/[E(0)I_{yy}(0)]$, $K_{zTR}L/[E(0)I_{yy}(0)]$, $K_{y\theta R}L/[E(0)I_{yy}(0)]$, $K_{yTR}L/[E(0)I_{yy}(0)]$
θ	= angle between principal and fixed frames
Λ^2	= dimensionless frequency, $m(0)\omega^2L^4/[E(0)I_{yy}(0)]$
ξ	= dimensionless distance to the left end of the beam, x/L
$\rho(x)$	= mass density per unit volume
σ_{xx}, ε_{xx}	= normal stress and strain in the x direction, respectively
τ	= dimensionless time
Φ	= pretwist angle of the beam
Ψ	= slope of the beam
ω	= circular frequency

Introduction

THE analysis of pretwisted beams is important in the design of a number of engineering components, e.g., turbine blades, helicopter rotor blades, and gear teeth. An interesting review of the subject can be found in Ref. 1. Approximation methods are very useful tools to investigate the vibrations of pretwisted blades, where exact solutions are difficult to obtain even for the simplest cases. An elastically restrained nonuniform beam with arbitrary pretwist is common in engineering applications. Therefore, it is neccessary to develop an accurate and simple method to solve the complicated problem.

For Bernoulli–Euler beams, Rosard and Lester[2] and Rao and Carnegie[3] used the transfer matrix method to determine the frequencies of vibration of the cantilever beam with uniform pretwist. Rosard and Lester[2] assumed that the displacements at each element are linear. Rao and Carnegie[3] used an iteration procedure to determine the displacements at each element while the initial displacements were assumed to be linear. Dawson[4] and Dawson and Carnegie[5] used the Rayleigh–Ritz method and transformation techniques to study the effects of uniform pretwist on the frequencies of cantilever blades. Carnegie and Thomas[6] and Rao[7–9] used the Rayleigh–Ritz method and Ritz–Galerkin method to study the effects of uniform pretwist and the taper ratio on the frequencies of cantilever blades, respectively. Sabuncu[10] presented the finite element method for the vibration analysis of pretwisted uniform cross-sectional blading. The variation of the pretwist along the blade length was assumed to be in linear or trigonometric increments.

For Timoshenko beams, the influence of the shear deformation and the rotatory inertia have been considered. Carnegie[11] determined the fundamental frequency of a cantilever beam by using Rayleigh's principle. Gupta and Rao[12] and Abbas[13] used the finite element method to determine the natural frequencies of uniformly pretwisted tapered cantilever blading. Subrahmanyam et al.[14] and Subrahmanyam and Rao[15] used the finite element method and the Reissner method to determined the natural frequencies of uniformly pretwisted tapered cantilever blading, respectively. Celep and Turhan[16] used the Galerkin method to investigate the influence of nonuniform pretwisting on the natural frequencies of uniform cross-sectional cantilever or simply supported beams. From the existing literature, it can be found that all of the previous investigations were restricted to cantilever or simply supported and tapered or uniform cross-sectional beams. There still is no study on the analysis of any nonuniform cross-sectional beam with arbitrary variation of pretwist angle and elastic boundary conditions.

In this paper, the governing differential equations with the elastic boundary conditions for the coupled bending–bending vibration of a pretwisted nonuniform beam are derived by using Hamilton's principle. A solution method, which is the generalization of the methods

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of Rosard and Lester[2] and Rao and Carnegie,[3] is presented. The field transfer matrix for the general system, instead of the approximate ones given by Rosard and Lester[2] and Rao and Carnegie,[3] is expressed in integration form. It is shown that, if the elements of the matrix can be integrated analytically, the exact matrix is obtained. The transfer matrix method is developed to determined the natural frequencies and the associated mode shapes of any nonuniform cross-sectional beam with arbitrary variation of pretwist angle. The frequency equation is derived and expressed in terms of the overall transfer matrix. Several limiting cases are examined. Finally, the influence of the elastically restrained boundary conditions and the uniform and nonuniform pretwists on the natural frequencies is also investigated.

Governing Equations and Boundary Conditions

Consider a pretwisted nonuniform beam with general elastic boundary conditions, as shown in Fig. 1. Neither shear deformation nor rotatory inertia are considered. The displacement fields of the beam are

$$u = -z\frac{\partial w}{\partial x} - y\frac{\partial v}{\partial x}, \qquad v = v(x,t), \qquad w = w(x,t) \tag{1}$$

The potential energy and the kinetic energy of the beam are expressed as follows:

$$\begin{aligned}\bar U &= \frac{1}{2}\int_0^L\int_A \sigma_{xx}\varepsilon_{xx}\,\mathrm{d}A\,\mathrm{d}x + \frac{1}{2}K_{yTL}v^2(0,t) \\ &+ \frac{1}{2}K_{yTR}v^2(L,t) + \frac{1}{2}K_{zTL}w^2(0,t) + \frac{1}{2}K_{zTR}w^2(L,t) \\ &+ \frac{1}{2}K_{y\theta L}\left[\frac{\partial v(0,t)}{\partial x}\right]^2 + \frac{1}{2}K_{y\theta R}\left[\frac{\partial v(L,t)}{\partial x}\right]^2 \\ &+ \frac{1}{2}K_{z\theta L}\left[\frac{\partial w(0,t)}{\partial x}\right]^2 + \frac{1}{2}K_{z\theta R}\left[\frac{\partial w(L,t)}{\partial x}\right]^2\end{aligned} \tag{2}$$

$$\bar K = \frac{1}{2}\int_0^L\left[\left(\frac{\partial w}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2\right]\rho A\,\mathrm{d}x \tag{3}$$

Application of Hamilton's principle yields the governing differential equations with the associated general elastic boundary conditions. The coupled governing differential equations are the same as those given by Rao.[7]

First, consider the free vibration of the beam. For harmonic vibration with circular frequency ω, the dimensionless governing characteristic differential equations of motion are written as

$$\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\left(B_{yy}\frac{\mathrm{d}^2W}{\mathrm{d}\xi^2}\right) + \frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\left(B_{yz}\frac{\mathrm{d}^2V}{\mathrm{d}\xi^2}\right) - \Lambda^2 mW = 0 \tag{4}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\left(B_{zz}\frac{\mathrm{d}^2V}{\mathrm{d}\xi^2}\right) + \frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\left(B_{yz}\frac{\mathrm{d}^2W}{\mathrm{d}\xi^2}\right) - \Lambda^2 mV = 0 \qquad \xi \in (0,1) \tag{5}$$

and the associated dimensionless general elastic boundary conditions are given as follows.

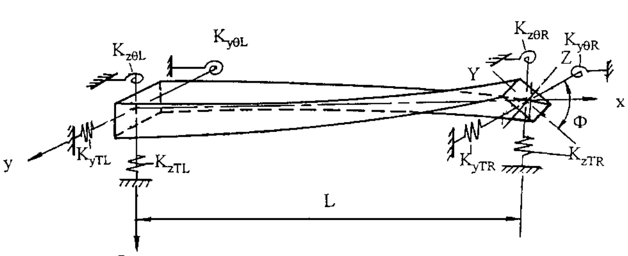

Fig. 1 Geometry and coordinate system of a pretwisted nonuniform beam with elastic boundary conditions.

At $\xi = 0$,

$$\gamma_{12}\left(B_{yy}\frac{\mathrm{d}^2W}{\mathrm{d}\xi^2} + B_{yz}\frac{\mathrm{d}^2V}{\mathrm{d}\xi^2}\right) - \gamma_{11}\frac{\mathrm{d}W}{\mathrm{d}\xi} = 0 \tag{6}$$

$$\gamma_{22}\left[\frac{\mathrm{d}}{\mathrm{d}\xi}\left(B_{yy}\frac{\mathrm{d}^2W}{\mathrm{d}\xi^2}\right) + \frac{\mathrm{d}}{\mathrm{d}\xi}\left(B_{yz}\frac{\mathrm{d}^2V}{\mathrm{d}\xi^2}\right)\right] + \gamma_{21}W = 0 \tag{7}$$

$$\gamma_{32}\left(B_{zz}\frac{\mathrm{d}^2V}{\mathrm{d}\xi^2} + B_{yz}\frac{\mathrm{d}^2W}{\mathrm{d}\xi^2}\right) - \gamma_{31}\frac{\mathrm{d}V}{\mathrm{d}\xi} = 0 \tag{8}$$

$$\gamma_{42}\left[\frac{\mathrm{d}}{\mathrm{d}\xi}\left(B_{zz}\frac{\mathrm{d}^2V}{\mathrm{d}\xi^2}\right) + \frac{\mathrm{d}}{\mathrm{d}\xi}\left(B_{yz}\frac{\mathrm{d}^2W}{\mathrm{d}\xi^2}\right)\right] + \gamma_{41}V = 0 \tag{9}$$

At $\xi = 1$,

$$\gamma_{52}\left(B_{yy}\frac{\mathrm{d}^2W}{\mathrm{d}\xi^2} + B_{yz}\frac{\mathrm{d}^2V}{\mathrm{d}\xi^2}\right) + \gamma_{51}\frac{\mathrm{d}W}{\mathrm{d}\xi} = 0 \tag{10}$$

$$\gamma_{62}\left[\frac{\mathrm{d}}{\mathrm{d}\xi}\left(B_{yy}\frac{\mathrm{d}^2W}{\mathrm{d}\xi^2}\right) + \frac{\mathrm{d}}{\mathrm{d}\xi}\left(B_{yz}\frac{\mathrm{d}^2V}{\mathrm{d}\xi^2}\right)\right] - \gamma_{61}W = 0 \tag{11}$$

$$\gamma_{72}\left(B_{zz}\frac{\mathrm{d}^2V}{\mathrm{d}\xi^2} + B_{yz}\frac{\mathrm{d}^2W}{\mathrm{d}\xi^2}\right) + \gamma_{71}\frac{\mathrm{d}V}{\mathrm{d}\xi} = 0 \tag{12}$$

$$\gamma_{82}\left[\frac{\mathrm{d}}{\mathrm{d}\xi}\left(B_{zz}\frac{\mathrm{d}^2V}{\mathrm{d}\xi^2}\right) + \frac{\mathrm{d}}{\mathrm{d}\xi}\left(B_{yz}\frac{\mathrm{d}^2W}{\mathrm{d}\xi^2}\right)\right] - \gamma_{81}V = 0 \tag{13}$$

where

$$\gamma_{i1} = \frac{\beta_i}{1+\beta_i}, \qquad \gamma_{i2} = \frac{1}{1+\beta_i} \tag{14}$$

Solution Method

The transfer matrix method is developed. The method is the generalization of the methods of Rosard and Lester[2] and Rao and Carnegie.[3] The closed-form field transfer matrix for the general system, instead of the approximate ones given by Rosard and Lester[2] and Rao and Carnegie,[3] is developed. It is assumed that the masses are concentrated in $n+1$ lumps at $n+1$ stations and the fields between the concentrated masses are massless but pose varying bending stiffnesses.

Closed-Form Field Transfer Matrix

Because the mass in the fields is neglected, the inertia terms in Eqs. (4) and (5) are not considered. In the jth field, the negative shear forces and moments are obtained by integrating the corresponding governing equations (4) and (5) once and twice, respectively:

$$-Q_z(\xi) = \frac{\mathrm{d}}{\mathrm{d}\xi}\left(B_{yy}\frac{\mathrm{d}^2W}{\mathrm{d}\xi^2}\right) + \frac{\mathrm{d}}{\mathrm{d}\xi}\left(B_{yz}\frac{\mathrm{d}^2V}{\mathrm{d}\xi^2}\right) = c_1 \qquad \xi \in (\xi_j, \xi_{j+1}) \tag{15}$$

$$-M_z(\xi) = B_{yy}\frac{\mathrm{d}^2W}{\mathrm{d}\xi^2} + B_{yz}\frac{\mathrm{d}^2V}{\mathrm{d}\xi^2} = c_1(\xi - \xi_j) + c_2 \qquad \xi \in (\xi_j, \xi_{j+1}) \tag{16}$$

$$-Q_y(\xi) = \frac{\mathrm{d}}{\mathrm{d}\xi}\left(B_{zz}\frac{\mathrm{d}^2V}{\mathrm{d}\xi^2}\right) + \frac{\mathrm{d}}{\mathrm{d}\xi}\left(B_{yz}\frac{\mathrm{d}^2W}{\mathrm{d}\xi^2}\right) = c_3 \qquad \xi \in (\xi_j, \xi_{j+1}) \tag{17}$$

$$-M_y(\xi) = B_{zz}\frac{\mathrm{d}^2V}{\mathrm{d}\xi^2} + B_{yz}\frac{\mathrm{d}^2W}{\mathrm{d}\xi^2} = c_3(\xi - \xi_j) + c_4 \qquad \xi \in (\xi_j, \xi_{j+1}) \tag{18}$$

Obviously, the coefficients c_1, c_2, c_3, and c_4 are the negative shear forces and the negative moments at $\xi = \xi_j$ in the direction of z and y, respectively. The following equations can be obtained easily from Eqs. (16) and (17):

$$\left(B_{zz}B_{yy} - B_{yz}^2\right)\frac{\mathrm{d}^2 W}{\mathrm{d}\xi^2} = c_1(\xi - \xi_j)B_{zz} + c_2 B_{zz} - c_3(\xi - \xi_j)B_{yz} - c_4 B_{yz} \tag{19}$$

$$\left(B_{yz}^2 - B_{zz}B_{yy}\right)\frac{\mathrm{d}^2 V}{\mathrm{d}\xi^2} = c_1(\xi - \xi_j)B_{yz} + c_2 B_{yz} - c_3(\xi - \xi_j)B_{yy} - c_4 B_{yy} \tag{20}$$

Integrating Eqs. (19) and (20) twice, respectively, one obtains

$$\frac{\mathrm{d}W}{\mathrm{d}\xi} = c_1\omega_1(\xi) + c_2\omega_2(\xi) + c_3\omega_3(\xi) + c_4\omega_4(\xi) + c_5 \tag{21}$$

$$W = c_1 w_1(\xi) + c_2 w_2(\xi) + c_3 w_3(\xi) + c_4 w_4(\xi) + c_5(\xi - \xi_j) + c_6 \tag{22}$$

$$\frac{\mathrm{d}V}{\mathrm{d}\xi} = c_1\upsilon_1(\xi) + c_2\upsilon_2(\xi) + c_3\upsilon_3(\xi) + c_4\upsilon_4(\xi) + c_7 \tag{23}$$

$$V = c_1 v_1(\xi) + c_2 v_2(\xi) + c_3 v_3(\xi) + c_4 v_4(\xi) + c_7(\xi - \xi_j) + c_8 \tag{24}$$

where c_5, c_6, c_7, and c_8 are $\mathrm{d}W(\xi_j)/\mathrm{d}\xi$, $W(\xi_j)$, $\mathrm{d}V(\xi_j)/\mathrm{d}\xi$, and $V(\xi_j)$, respectively, and

$$\omega_1(\xi) = \int_{\xi_j}^{\xi} \frac{(\zeta - \xi_j)B_{zz}(\zeta)}{B_{zz}(\zeta)B_{yy}(\zeta) - B_{yz}^2(\zeta)}\,\mathrm{d}\zeta$$

$$\omega_2(\xi) = \int_{\xi_j}^{\xi} \frac{B_{zz}(\zeta)}{B_{zz}(\zeta)B_{yy}(\zeta) - B_{yz}^2(\zeta)}\,\mathrm{d}\zeta \tag{25a}$$

$$\omega_3(\xi) = \int_{\xi_j}^{\xi} \frac{(\zeta - \xi_j)B_{yz}(\zeta)}{B_{yz}^2(\zeta) - B_{zz}(\zeta)B_{yy}(\zeta)}\,\mathrm{d}\zeta$$

$$\omega_4(\xi) = \int_{\xi_j}^{\xi} \frac{B_{yz}(\zeta)}{B_{yz}^2(\zeta) - B_{zz}(\zeta)B_{yy}(\zeta)}\,\mathrm{d}\zeta$$

$$w_i(\xi) = \int_{\xi_j}^{\xi} \omega_i(\zeta)\,\mathrm{d}\zeta, \qquad i = 1, 2, 3, 4 \tag{25b}$$

$$\upsilon_1(\xi) = \int_{\xi_j}^{\xi} \frac{(\zeta - \xi_j)B_{yz}(\zeta)}{B_{yz}^2(\zeta) - B_{zz}(\zeta)B_{yy}(\zeta)}\,\mathrm{d}\zeta$$

$$\upsilon_2(\xi) = \int_{\xi_j}^{\xi} \frac{B_{yz}(\zeta)}{B_{yz}^2(\zeta) - B_{zz}(\zeta)B_{yy}(\zeta)}\,\mathrm{d}\zeta \tag{25c}$$

$$\upsilon_3(\xi) = \int_{\xi_j}^{\xi} \frac{(\zeta - \xi_j)B_{yy}(\zeta)}{B_{zz}(\zeta)B_{yy}(\zeta) - B_{yz}^2(\zeta)}\,\mathrm{d}\zeta$$

$$\upsilon_4(\xi) = \int_{\xi_j}^{\xi} \frac{B_{yy}(\zeta)}{B_{zz}(\zeta)B_{yy}(\zeta) - B_{yz}^2(\zeta)}\,\mathrm{d}\zeta$$

$$v_i(\xi) = \int_{\xi_j}^{\xi} \upsilon_i(\zeta)\,\mathrm{d}\zeta, \qquad i = 1, 2, 3, 4 \tag{25d}$$

The field transfer matrix relation with arbitrarily varying coefficients is then obtained from Eqs. (15–25):

$$\left[w_{j+1}^L \;\; \Psi_{z,j+1}^L \;\; M_{z,j+1}^L \;\; Q_{z,j+1}^L \;\; v_{j+1}^L \;\; \Psi_{y,j+1}^L \;\; M_{y,j+1}^L \;\; Q_{y,j+1}^L\right]^T = \left[T_f\right]_j \left[w_j^R \;\; \Psi_{z,j}^R \;\; M_{z,j}^R \;\; Q_{z,j}^R \;\; v_j^R \;\; \Psi_{y,j}^R \;\; M_{y,j}^R \;\; Q_{y,j}^R\right]^T \tag{26}$$

where the field transfer matrix pertaining to the jth field $[T_f]_j$ is

$$[T_f]_j = \begin{bmatrix} 1 & \xi & -w_2(\xi) & -w_1(\xi) & 0 & 0 & -w_4(\xi) & -w_3(\xi) \\ 0 & 1 & -\omega_2(\xi) & -\omega_1(\xi) & 0 & 0 & -\omega_4(\xi) & -\omega_3(\xi) \\ 0 & 0 & 1 & \xi & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -v_2(\xi) & -v_1(\xi) & 1 & \xi & -v_4(\xi) & -v_3(\xi) \\ 0 & 0 & -\upsilon_2(\xi) & -\upsilon_1(\xi) & 0 & 1 & -\upsilon_4(\xi) & -\upsilon_3(\xi) \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \xi \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{27}$$

It is shown that, if the elements of the matrix in integration form can be integrated analytically, the exact matrix is obtained. Otherwise, a semiexact solution can be easily obtained by using the numerical integration method. Obviously, if the beam is unpretwisted and of uniform cross section, the field transfer matrix can be uncoupled into two terms in the displacements W and V, respectively. These uncoupled results are the same as those given by Meirovitch.[17]

Station Transfer Matrix

The beam is divided into n number of elements of equal length $1/n$. The mass is assumed to be concentrated in $n+1$ lumps at $n+1$ stations. Equations (4) and (5) can be written as

$$\frac{\mathrm{d}Q_z}{\mathrm{d}\xi} = -\Lambda^2 m(\xi)W(\xi) \tag{28}$$

$$\frac{\mathrm{d}Q_y}{\mathrm{d}\xi} = -\Lambda^2 m(\xi)V(\xi) \tag{29}$$

The distribution of the mass is determined by statics.[18] Applying a numerical scheme in Eqs. (28) and (29) yields

$$Q_{z,j}^R = Q_{z,j}^L - \Lambda^2 m_j W_j \tag{30}$$

$$Q_{y,j}^R = Q_{y,j}^L - \Lambda^2 m_j V_j \tag{31}$$

where

$$m_1 = n\int_0^{\xi_2}(\xi_2 - \zeta)m(\zeta)\,\mathrm{d}\zeta$$

$$m_j = n\int_{\xi_{j-1}}^{\xi_j}\left(\zeta - \xi_{j-1}\right)m(\zeta)\,\mathrm{d}\zeta + n\int_{\xi_j}^{\xi_{j+1}}(\xi_{j+1} - \zeta)m(\zeta)\,\mathrm{d}\zeta \qquad j = 2, \ldots, n \tag{32}$$

$$m_{n+1} = n\int_{\xi_n}^{1}\left(\zeta - \xi_n\right)m(\zeta)\,\mathrm{d}\zeta$$

The continuity conditions are

$$W_j^R = W_j^L = W_j, \qquad V_j^R = V_j^L = V_j$$
$$\Psi_{z,j}^R = \Psi_{z,j}^L = \Psi_{z,j}, \qquad \Psi_{y,j}^R = \Psi_{y,j}^L = \Psi_{y,j} \tag{33}$$
$$M_{z,j}^R = M_{z,j}^L, \qquad M_{y,j}^R = M_{y,j}^L$$

Using Eqs. (30–33), the station transfer matrix relation is obtained as

$$\begin{Bmatrix} W_j^R \\ \Psi_{z,j}^R \\ M_{z,j}^R \\ Q_{z,j}^R \\ v_j^R \\ \Psi_{y,j}^R \\ M_{y,j}^R \\ Q_{y,j}^R \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -\Lambda^2 m_j & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -\Lambda^2 m_j & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} W_j^L \\ \Psi_{z,j}^L \\ M_{z,j}^L \\ Q_{z,j}^L \\ v_j^L \\ \Psi_{y,j}^L \\ M_{y,j}^L \\ Q_{y,j}^L \end{Bmatrix} \tag{34}$$

The first square matrix on the right-hand side of Eq. (34) is the station transfer matrix pertaining to the jth station, denoted $[T_s]_j$. The transfer matrix $[T]_j$ for the jth station and field is the product of $[T_f]_j$ and $[T_s]_j$. It can be obtained as

$$
[T]_j = \begin{bmatrix} 1+\Lambda^2 m_j w_1(\xi) & \xi & -w_2(\xi) & -w_1(\xi) & \Lambda^2 m_j w_3 & 0 & -w_4(\xi) & -w_3(\xi) \\ \Lambda^2 m_j \omega_1(\xi) & 1 & -\omega_2(\xi) & -\omega_1(\xi) & \Lambda^2 m_j \omega_3 & 0 & -\omega_4(\xi) & -\omega_3(\xi) \\ -\Lambda^2 m_j \xi & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -\Lambda^2 m_j & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \Lambda^2 m_j v_1(\xi) & 0 & -v_2(\xi) & -v_1(\xi) & 1+\Lambda^2 m_j v_3(\xi) & \xi & -v_4(\xi) & -v_3(\xi) \\ \Lambda^2 m_j \upsilon_1(\xi) & 0 & -\upsilon_2(\xi) & -\upsilon_1(\xi) & \Lambda^2 m_j \upsilon_3(\xi) & 1 & -\upsilon_4(\xi) & -\upsilon_3(\xi) \\ 0 & 0 & 0 & 0 & -\Lambda^2 m_j \xi & 0 & 1 & \xi \\ 0 & 0 & 0 & 0 & -\Lambda^2 m_j & 0 & 0 & 1 \end{bmatrix} \tag{35}
$$

Frequency Equation

According to the preceding results, the overall transfer matrix relation is obtained as

$$
\begin{bmatrix} w^R_{n+1} & \Psi^R_{z,n+1} & M^R_{z,n+1} & Q^R_{z,n+1} & v^R_{n+1} & \Psi^R_{y,n+1} & M^R_{y,n+1} & Q^R_{y,n+1} \end{bmatrix}^T
$$

$$
= [T]\begin{bmatrix} w^L_1 & \Psi^L_{z,1} & M^L_{z,1} & Q^L_{z,1} & v^L_1 & \Psi^L_{y,1} & M^L_{y,1} & Q^L_{y,1} \end{bmatrix}^T \tag{36}
$$

where the overall transfer matrix $[T]$ can be expressed as

$$
[T] = [T_s]_{n+1} \prod_{i=n}^{1} [T]_i \tag{37}
$$

The state vectors at the first and the $(n+1)$th stations are the states at the left and the right ends of the beam, respectively. Substituting Eqs. (34–37) into the boundary conditions (6–14), the frequency equation of the general system is obtained as

$$
\begin{aligned}
&-\gamma_{11}\gamma_{21}[-\gamma_{31}\gamma_{41}D_1(3,4) + \gamma_{31}\gamma_{42}D_2(3,4) \\
&+ \gamma_{32}\gamma_{41}D_3(3,4) - \gamma_{32}\gamma_{42}D_4(3,4)] \\
&+ \gamma_{11}\gamma_{22}[-\gamma_{31}\gamma_{41}D_1(1,3) + \gamma_{31}\gamma_{42}D_2(1,3) \\
&+ \gamma_{32}\gamma_{41}D_3(1,3) - \gamma_{32}\gamma_{42}D_4(1,3)] \\
&+ \gamma_{12}\gamma_{21}[-\gamma_{31}\gamma_{41}D_1(2,4) + \gamma_{31}\gamma_{42}D_2(2,4) \\
&+ \gamma_{32}\gamma_{41}D_3(2,4) - \gamma_{32}\gamma_{42}D_4(2,4)] \\
&- \gamma_{12}\gamma_{22}[-\gamma_{31}\gamma_{41}D_1(1,2) + \gamma_{31}\gamma_{42}D_2(1,2) \\
&+ \gamma_{32}\gamma_{41}D_3(1,2) - \gamma_{32}\gamma_{42}D_4(1,2)] = 0
\end{aligned} \tag{38}
$$

where

$$
D_1(i,j) = \begin{vmatrix} \beta_{1i} & \beta_{1j} & \beta_{17} & \beta_{18} \\ \beta_{2i} & \beta_{2j} & \beta_{27} & \beta_{28} \\ \beta_{3i} & \beta_{3j} & \beta_{37} & \beta_{38} \\ \beta_{4i} & \beta_{4j} & \beta_{47} & \beta_{48} \end{vmatrix}
$$

$$
D_2(i,j) = \begin{vmatrix} \beta_{1i} & \beta_{1j} & \beta_{15} & \beta_{17} \\ \beta_{2i} & \beta_{2j} & \beta_{25} & \beta_{27} \\ \beta_{3i} & \beta_{3j} & \beta_{35} & \beta_{37} \\ \beta_{4i} & \beta_{4j} & \beta_{45} & \beta_{47} \end{vmatrix}
\tag{39}
$$

$$
D_3(i,j) = \begin{vmatrix} \beta_{1i} & \beta_{1j} & \beta_{16} & \beta_{18} \\ \beta_{2i} & \beta_{2j} & \beta_{26} & \beta_{28} \\ \beta_{3i} & \beta_{3j} & \beta_{36} & \beta_{38} \\ \beta_{4i} & \beta_{4j} & \beta_{46} & \beta_{48} \end{vmatrix}
$$

$$
D_4(i,j) = \begin{vmatrix} \beta_{1i} & \beta_{1j} & \beta_{15} & \beta_{16} \\ \beta_{2i} & \beta_{2j} & \beta_{25} & \beta_{26} \\ \beta_{3i} & \beta_{3j} & \beta_{35} & \beta_{36} \\ \beta_{4i} & \beta_{4j} & \beta_{45} & \beta_{46} \end{vmatrix}
$$

in which

$$
\begin{aligned}
\beta_{1,j} &= -\gamma_{52}T_{3j} + \gamma_{51}T_{2j} \\
\beta_{2,j} &= \gamma_{62}T_{4j} + \gamma_{61}T_{1j} \\
\beta_{3,j} &= -\gamma_{72}T_{7j} + \gamma_{71}T_{6j} \\
\beta_{4,j} &= \gamma_{82}T_{8j} + \gamma_{81}T_{5j}, \qquad j = 1, 2, \ldots, 7, 8
\end{aligned} \tag{40}
$$

The frequency equations of four limiting cases are tabulated in the Appendix.

Numerical Results and Discussion

To illustrate the application of the method, to compare the results with those in the existing literature, and to explore the physical phenomena of the system, the three following examples are presented.

To demonstrate efficiency and convergence of the proposed numerical method, the first, second, and third frequencies are determined for a cantilever uniform pretwisted beam. For example 1, Fig. 2 shows the convergence of the frequencies as a function of the number of subsections is increased. For comparison, the natural frequencies obtained by the proposed method, as well as those given by other investigators,[3,7,10,14] are given in Table 1. Excellent agreement is obtained between the previous numerical and experimental results and those by the proposed method.

For example 2, Fig. 3a shows the influence of the spring constants β_5, β_6, β_7, and β_8 on the natural frequencies of a uniform

Table 1 Frequencies of a cantilever pretwisted uniform beam [$\Phi = \pi/4, I_{YY}/I_{ZZ} = 0.004624$]

	Rayleigh–Ritz method[3]	Galerkin process[7]	Reissner method[14]	Experimental values[10]	Present study
Λ_1	3.536	3.530	3.540	3.365	3.478
Λ_2	17.17	17.40	17.41	16.54	17.11
Λ_3	54.36	54.13	54.26	52.47	53.53
Λ_4	70.15	69.58	69.71	63.31	69.90

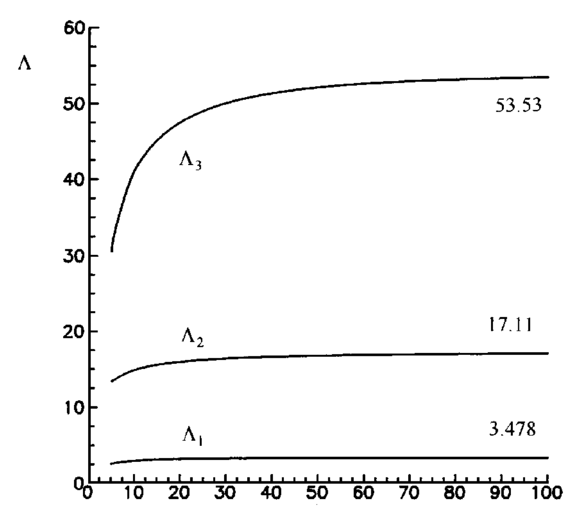

Fig. 2 Convergence of the first, second, and third frequencies for a cantilever pretwisted uniform beam [$\Phi = \pi/4, I_{YY}/I_{ZZ} = 0.004624$].

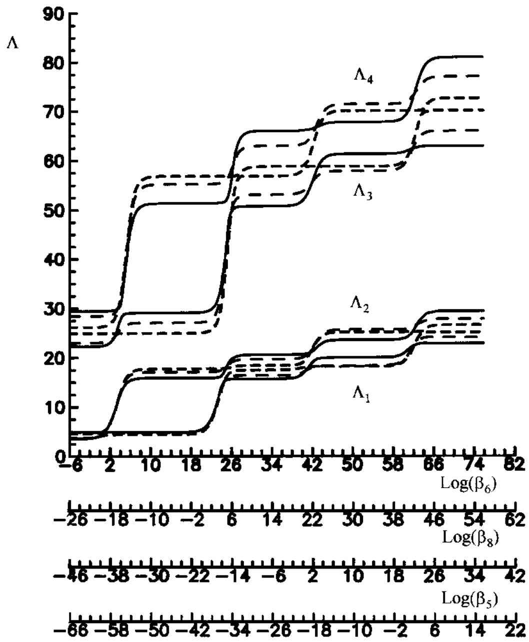

Fig. 3a Influence of the spring constants β_5, β_6, β_7, and β_8 on the frequencies; $\beta_1, \beta_2, \beta_3, \beta_4 \to \infty$, I_{YY}/I_{ZZ} = 0.5: ——, $\Phi = \pi/4$; - - -, $\Phi = \pi/2$; and - - - -, $\Phi = \pi$.

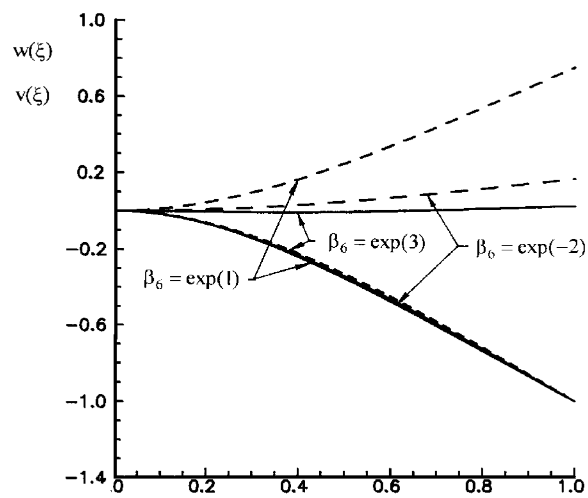

Fig. 3b First mode shape; $\beta_1, \beta_2, \beta_3, \beta_4 \to \infty$ and $\beta_5 = \beta_7 = \beta_8 = 0$, I_{YY}/I_{ZZ} = 0.5, $\Phi = \pi/4$: ——, $w(\xi)$ and - - - -, $v(\xi)$.

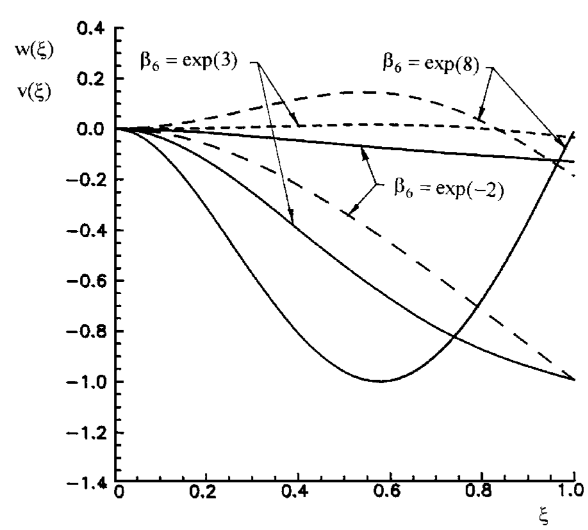

Fig. 3c Second mode shape; $\beta_1, \beta_2, \beta_3, \beta_4 \to \infty$ and $\beta_5 = \beta_7 = \beta_8 = 0$, I_{YY}/I_{ZZ} = 0.5, $\Phi = \pi/4$: ——, $w(\xi)$ and - - - -, $v(\xi)$.

beam clamped at the left end, pretwisted uniformly along its length by an angle Φ. It is true that, when a spring constant is decreased to a certain value, the natural frequencies will approach constant values. This means that the spring constant can be seen to be zero in this situation. Similarly, when a spring constant is increased to a certain value, the natural frequencies will approach constant values. This means that the spring constant can be seen to be infinite in this situation. Based on this and with the intention of simultaneously studying the effect of the four spring constants on the natural frequencies, one assumes $\beta_6 = \beta_8 \exp(20) = \beta_5 \exp(40) = \beta_7 \exp(60)$. When the translational spring constant β_6 is decreased to less than the value of $\exp(-2)$, the natural frequencies will approach constant values, and the system becomes a clamped–free system. When β_6 is increased to the value of $\exp(-2)$, the natural frequencies evidently increase. The natural frequencies will all approach constant values as β_6 is increased to the value of exp(12), and the system becomes a clamped–simply supported system in the z direction. When the translational spring constant β_8 is increased to the value of $\exp(-2)$, the natural frequencies evidently increase. The natural frequencies will all approach constant values as β_8 is increased to the value of exp(12), and the system becomes a clamped–simply supported system in both the z and y directions. When the rotational spring constant β_5 is increased to the value of $\exp(-2)$, the natural frequencies evidently increase. The natural frequencies will all approach constant values as β_5 is increased to the value of exp(10), and the system becomes clamped and hinged in the z and y directions, respectively, at the right end. When the rotational spring constant β_7 is increased to the value of $\exp(-2)$, the natural frequencies evidently increase. The natural frequencies will all approach constant values as β_7 is increased to the value of exp(8), and the system becomes a clamped–clamped system. It is observed that the effect of a spring constant on the natural frequencies of a soft system is greater than the effect on a stiff system.

The translational spring constant β_6 has almost no influence on the third natural frequency of a beam with pretwist angle of π. The influence of the translational spring constant β_6 on the first frequency is small, but large on the second frequency. The reason can be found in Figs. 3b and 3c: when the translational spring constant is relatively low, the first and second mode shapes are dominant in the displacements w and v, respectively. Thus, the influence of the translational spring constant on the first frequency is greater than on the second frequency. But when the translational spring constant is increased, the first and second mode shapes are dominant in the displacements v and w, respectively. Thus, the influence of the translational spring constant on the second frequency is greater than on the first frequency. It can be observed in Figs. 3b and 3c that, when the translational spring constant is relatively high, the

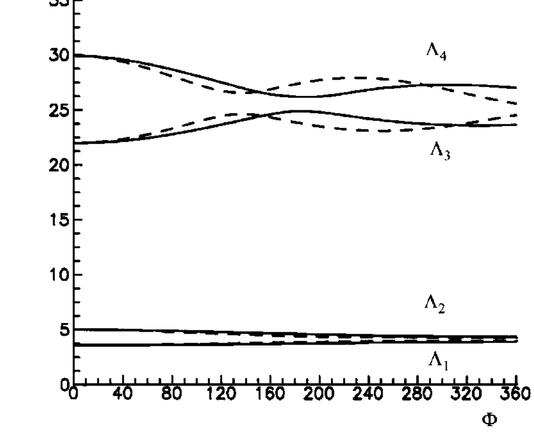

Fig. 4 Influence of the pretwist angle Φ on the frequencies; $\beta_1, \beta_2, \beta_3, \beta_4 \to \infty$, $\beta_5 = \beta_6 = \beta_7 = \beta_8 = 0$, $B_{yy} = (1 - 0.1\xi)\cos^2\theta + 2(1 - 0.1\xi)^3 \sin^2\theta$, $B_{zz} = 2(1 - 0.1\xi)^3 \cos^2\theta + (1 - 0.1\xi)\sin^2\theta$, and $B_{yz} = [(1 - 0.1\xi)^3 - 0.5(1 - 0.1\xi)]\sin^2\theta$: ——, $\theta = \xi\Phi$ and - - -, $\theta = \Phi \sin(\pi\xi/2)$.

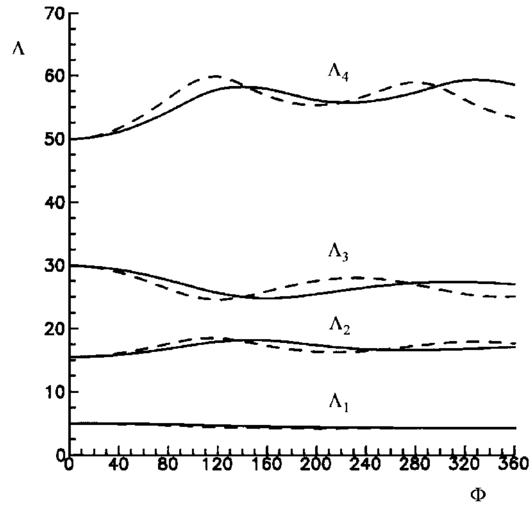

Fig. 5 Influence of the pretwist angle Φ on the frequencies; $\beta_1, \beta_2, \beta_3, \beta_4, \beta_6 \to \infty$, $\beta_5 = \beta_7 = \beta_8 = 0$, $B_{yy} = (1 - 0.1\xi)\cos^2\theta + 2(1 - 0.1\xi)^3 \sin^2\theta$, $B_{zz} = 2(1 - 0.1\xi)^3\cos^2\theta + (1 - 0.1\xi)\sin^2\theta$, and $B_{yz} = [(1 - 0.1\xi)^3 - 0.5(1 - 0.1\xi)]\sin^2\theta$: ——, $\theta = \xi\Phi$ and - - -, $\theta = \Phi\sin(\pi\xi/2)$.

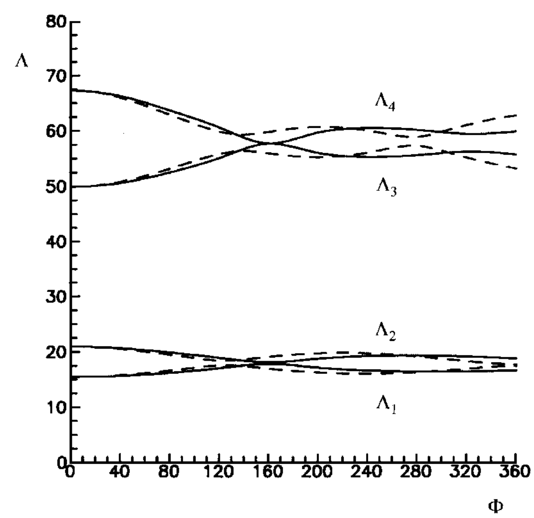

Fig. 6 Influence of the pretwist angle Φ on the frequencies; $\beta_1, \beta_2, \beta_3, \beta_4, \beta_6, \beta_8 \to \infty$, $\beta_5 = \beta_7 = 0$, $B_{yy} = (1 - 0.1\xi)\cos^2\theta + 2(1 - 0.1\xi)^3 \sin^2\theta$, $B_{zz} = 2(1 - 0.1\xi)^3\cos^2\theta + (1 - 0.1\xi)\sin^2\theta$, and $B_{yz} = [(1 - 0.1\xi)^3 - 0.5(1 - 0.1\xi)]\sin^2\theta$: ——, $\theta = \xi\Phi$ and - - -, $\theta = \Phi\sin(\pi\xi/2)$.

displacement $w(1)$ approaches zero. Hence, the translational spring constant has almost no influence on the natural frequencies. It can be concluded that the effects of the spring constants on the natural frequencies are decided on the associated mode shapes.

For example 3, Figs. 4–6 show the influence of the pretwist angle Φ on the natural frequencies of three kinds of tapered beams with uniform and nonuniform pretwist. Figure 4 shows the results of a cantilever beam, Fig. 5 is for a clamped–simply supported in the z direction beam, and Fig. 6 is for a clamped–hinged beam. It can be observed that the influence of the pretwist angle on the natural frequencies is periodic. The influence on the natural frequencies of higher modes is greater than that on the natural frequencies of lower modes. The stiffer the boundary supports are, the greater the influence. The influence of the pretwist angle on the natural frequencies of the beam with nonuniform pretwist is greater than that of the beam with uniform pretwist.

Summary

The governing differential equations with the general elastic boundary conditions for the coupled bending–bending vibration of a pretwist nonuniform beam are derived by using Hamilton's principle. The field transfer matrix for the general system is expressed in integration form. It is shown that, if the elements of the matrix can be integrated analytically, the exact matrix is obtained. The transfer matrix method is developed to study the vibrations of any nonuniform cross-sectional beam with arbitrary variation of pretwist angle. The influence of the pretwist angle on the natural frequencies is periodic. The influence of the pretwist angle on the natural frequencies of higher modes is greater than that on the natural frequencies of lower modes. The stiffer the boundary supports are, the greater the influence of the pretwist angle on the natural frequencies. The influence of the pretwist angle on the natural frequencies of the beam with nonuniform pretwist is greater than that of the beam with uniform pretwist.

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Appendix: Frequency Equations

Case 1: Clamped–Clamped

For this case, $\gamma_{i1} = 1$ and $\gamma_{i2} = 0$, and the frequency equation is

$$\begin{vmatrix} T_{13} & T_{14} & T_{17} & T_{18} \\ T_{23} & T_{24} & T_{27} & T_{28} \\ T_{53} & T_{54} & T_{57} & T_{58} \\ T_{63} & T_{64} & T_{67} & T_{68} \end{vmatrix} = 0 \tag{A1}$$

where T_{ij} are the elements of the overall transfer matrix (37).

Case 2: Clamped–Free

For this case, $\gamma_{11} = \gamma_{21} = \gamma_{31} = \gamma_{41} = \gamma_{52} = \gamma_{62} = \gamma_{72} = \gamma_{82} = 1$ and $\gamma_{12} = \gamma_{22} = \gamma_{32} = \gamma_{42} = \gamma_{51} = \gamma_{61} = \gamma_{71} = \gamma_{81} = 0$, and the frequency equation is

$$\begin{vmatrix} T_{33} & T_{34} & T_{37} & T_{38} \\ T_{43} & T_{44} & T_{47} & T_{48} \\ T_{73} & T_{74} & T_{77} & T_{78} \\ T_{83} & T_{84} & T_{87} & T_{88} \end{vmatrix} = 0 \tag{A2}$$

where T_{ij} are the elements of the overall transfer matrix (37).

Case 3: Hinged–Hinged

For this case, $\gamma_{12} = \gamma_{21} = \gamma_{32} = \gamma_{41} = \gamma_{52} = \gamma_{61} = \gamma_{72} = \gamma_{81} = 1$ and $\gamma_{11} = \gamma_{22} = \gamma_{31} = \gamma_{42} = \gamma_{51} = \gamma_{62} = \gamma_{71} = \gamma_{82} = 0$, and the frequency equation is

$$\begin{vmatrix} T_{12} & T_{14} & T_{16} & T_{18} \\ T_{32} & T_{34} & T_{36} & T_{38} \\ T_{52} & T_{54} & T_{56} & T_{58} \\ T_{72} & T_{74} & T_{76} & T_{78} \end{vmatrix} = 0 \tag{A3}$$

where T_{ij} are the elements of the overall transfer matrix (37).

Case 4: Clamped–Spring

For this case, $\gamma_{11} = \gamma_{21} = \gamma_{31} = \gamma_{41} = 1$ and $\gamma_{12} = \gamma_{22} = \gamma_{32} = \gamma_{42} = 0$, and the frequency equation is

$$\begin{vmatrix} \beta_{13} & \beta_{14} & \beta_{17} & \beta_{18} \\ \beta_{23} & \beta_{24} & \beta_{27} & \beta_{28} \\ \beta_{33} & \beta_{34} & \beta_{37} & \beta_{38} \\ \beta_{43} & \beta_{44} & \beta_{47} & \beta_{48} \end{vmatrix} = 0 \tag{A4}$$

where

$$\begin{aligned} \beta_{1j} &= \gamma_{51} T_{2j} - \gamma_{52} T_{3j}, \qquad & \beta_{2j} &= \gamma_{62} T_{4j} + \gamma_{61} T_{1j} \\ \beta_{3j} &= \gamma_{71} T_{6j} - \gamma_{72} T_{7j}, \qquad & \beta_{4j} &= \gamma_{81} T_{5j} + \gamma_{82} T_{8j} \end{aligned} \tag{A5}$$

in which T_{ij} are the elements of the overall transfer matrix (37).

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